A Semi-Algorithmic Search for Lie Symmetries

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Abstract

In [1] we defined a function (we called S) associated to a rational second order ordinary differential equation (rational 2ODE) that is linked to the search of an integrating factor. In this work we investigate the relation between these S-functions and the Lie symmetries of a rational 2ODE. Based on this relation we can construct a semi-algorithmic method to find the Lie symmetries of a 2ODE even in the case where it presents no Lie point symmetries.

Keyword: Lie Symmetry, Second Order Ordinary Differential Equations, S-function, Darboux Polynomials

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1 Introduction

Lie group analysis is perhaps the most powerful tool to study differential equations (DEs) [2, 3, 4, 5, 6, 7, 8, 9, 10]. S. Lie [2] demonstrated that the majority of the technics of solving differential equations could be unified under the same theoretical background: the invariance of the DEs being solved under a continuous group of transformations (a Lie group). Since its appearance in the end of the 19th century, the Lie analysis of DEs has suffered a fantastic grown up especially in the last decades. One of the main reasons is, probably, the appearance of the computer in the second half of the 20th century, since it turns long and complicated symbolic calculations into the simple pushing of a button. Unfortunately, in real life, nothing is that simple. For the computer have the job done, we need to 'tell' it exactly what to do. In other words, it is necessary to furnish an algorithm, i.e., a finite sequence of determined steps. However, in Lies method, there is no algorithmic procedure to solve the determining equations for the *infinitesimals* (i.e., the coefficients of the symmetry generators)[2]. The things go worse when we are dealing with an ODE not possessing Lie point symmetries (presenting only dynamical symmetries). In this case we can not separate the determining equation in the derivatives and, instead of determining equations, we have only one determining equation: a 'very ugly' partial differential equation (PDE) that may leave us completely lost. In this last situation, we can not count even with a sistematic way to search for the symmetries; and, witout the symmetries, the Lie method can not be applied.

To overcome the difficulties in the treatment of ODEs (or systems of ODEs) that do not possess Lie point symmetries several approaches have been developed: P. J. Olver introduced the concept of *exponencial vector field* (see [3], p. 185); B. Abraham-Shrauner, A. Guo, K.S. Govinder, P.G.L. Leach, F.M. Mahomed, A.A. Adam and others worked with the concept of *hiden* and *non local symetries* [13, 14, 15, 16, 17, 18]. C. Muriel and J.L. Romero have developed the concept of λ-*symmetry* [19, 20] and E. Pucci and G. Saccomandi created the concept of *telescopical symmetry* [21]. Another great approach was brought by M.C. Nucci by making use of the *Jacobi last multiplier* [22].

Despite all these developments, there is still no fully algorithmic method to solve the determining equations. This mean that, in some part of the whole process, we can face a set of PDEs (the determining equations) for the infinitesimals we don't know how to deal with.

In [1] we have developed an extension of the Prelle-Singer method [23] and, in that paper, we have proposed to use an unknown function (that we called S [3]) in order to make the 1-form[4] $\phi(x,y,y')\,dx - dy'$ proportional to an exact 1-form. In [27], we constructed a semi-algorithm to determine the S-function for an 2ODE presenting an elementary[5] first integral. In this paper we study the relation between the S-functions and the Lie symmetries of a rational 2ODE. Based on this relation we propose a semi-algorithm to calculate the Lie symmetries for rational 2ODEs. This procedure can succeed even in the case where there are only dynamical

[2]Many computer packages implement some *heuristics* to find the infinitesimals and apply the Lie method. See, for example, [11, 12].

[3]This idea was further pursued by [24, 25, 26].

[4]This 1-form is associated with the rational 2ODE $y'' = \phi(x,y,y')$, where ϕ is a rational function of (x,y,y').

[5]For a formal definition of elementary function, please see [28].

symmetries (i.e., no Lie point symmetries).

In this work the main idea is to present a connection between a Darboux type method and the search for Lie symmetries of 2ODEs. Here, we present one such algorithm that we expect is the firt of a series of developments in that area.

The paper is organized as follows: In section 2, we present some basic definitions and stablish the relation between the S-functions and the Lie symmetries of a 2ODE. In section 3, we propose a method (a semi-algorithm) to calculate a Lie symmetry *associated* with an S-function. In section 4, we present two examples to show the method in action. Finally, we present our conclusions and point out some directions to further our work.

2 Lie Symmetries and S-functions

In this section, we will describe how the S-function is defined and the relation between the Lie symmetries and the S-functions of a 2ODE.

2.1 The definition of the S-functions

Let us consider the 2ODE given by:

$$y'' = \phi(x, y, y'), \tag{1}$$

where ϕ is a function of (x, y, y').

A function $I(x, y, y')$ defines a first integral (conserved quantity) of (1) if $I(x, y, y')$ is constant over all solution curves of (1). So, in other words, the 1-form defined by $\omega \equiv dI = I_x\, dx + I_y\, dy + I_{y'}\, dy'$ is null if the 1-forms $(\alpha,\, \beta)$ defined by:

$$\begin{aligned} \alpha &= \phi\, dx - dy' \\ \beta &= y' dx - dy \end{aligned} \tag{2}$$

are null. This implies that

$$\omega = dI = r\alpha \,+\, s\beta \tag{3}$$

where r and s are functions of (x, y, y'). Thus

$$dI = I_x dx + I_y dy + I_{y'} dy = r(\phi\, dx - dy') \,+\, s(y' dx - dy) \tag{4}$$

implying that $I_x = r\,\phi + s\,y'$, $I_y = -s$ and $I_{y'} = -r$. Therefore, if we determine r and s, we can find I via quadratures (see [29, 30]).

Let us make some definitions that make it easier to show some connection between the coeficients r and s and the symmetries of the 2ODE (1).

Definition 1: *A function $R(x, y, y')$ satisfying*

$$R\,(A\, dx + B\, dy + C\, dy') = d\,\gamma, \tag{5}$$

where A, B and C are functions of (x, y, y') and $d\,\gamma$ is an exact 1-form, is called an integrating factor for the 1-form $(A\, dx + B\, dy + C\, dy')$.

Definition 2: *Let $S(x,y,y')$ be a function defined by*

$$S \equiv \frac{s}{r}, \tag{6}$$

where r and s are functions satisfying (3). We will call it a S-function associated with the 2ODE (1).

Rewriting (4), we have

$$dI = I_x dx + I_y dy + I_{y'} dy = r\left((\phi + \frac{s}{r}\, y')dx - \frac{s}{r}\, dy - dy' \right). \tag{7}$$

We can see that r is an integrating factor for the 1-form $\left((\phi + \frac{s}{r}\, y')dx - \frac{s}{r}\, dy - dy'\right)$. So, writing $r = R$ and using the definition for the S-function we can finally write

$$dI = R\left((\phi + \, S\, y')dx - S\, dy - dy'\right). \tag{8}$$

2.2 The connection of the S-functions with the Lie symmetries of a 2ODE

We will begin this section by stating a theorem (our first goal):

Theorem 1: *Let $y'' = \phi(x,y,y')$ be a 2ODE presenting a first integral $I(x,y,y')$. If $\overline{\eta}(x,y,y')$ is the infinitesimal of a Lie symmetry generator in the evolutionary form, then the function S defined by*

$$S \equiv -\frac{D_x[\overline{\eta}]}{\overline{\eta}}, \tag{9}$$

where D_x is the operator defined by

$$D_x \equiv \partial_x + y'\, \partial_y + \phi(x,y,y')\, \partial_{y'}\,, \tag{10}$$

is a S-function associated with the 2ODE $y'' = \phi(x,y,y')$.

To prove this result we will use the following lemma:

Lemma 1: *Let $y'' = \phi(x,y,y')$ be a 2ODE. If $\overline{\eta}(x,y,y')$ is the infinitesimal of a Lie symmetry generator in the evolutionary form, then $\overline{\eta}$ must obey the following PDE:*

$$D_x^2[\overline{\eta}] = D_x[\overline{\eta}]\, \phi_{y'} + \overline{\eta}\, \phi_y, \tag{11}$$

where D_x is the operator defined by (10).

Proof of Lemma 1: If the hypothesis of the lemma is fulfilled, then

$$X^{(2)}[y'' - \phi(x,y,y')] = 0, \tag{12}$$

where $X^{(2)} = \xi\, \partial_x + \eta\, \partial_y + \eta^{(1)}\, \partial_{y'} + \eta^{(2)}\, \partial_{y''}$ is the second extension of the group generator $X = \xi\, \partial_x + \eta\, \partial_y$ and $\eta^{(1)}$ and $\eta^{(2)}$ are given by

$$\begin{aligned} \eta^{(1)} &\equiv D_x[\eta] - y'\, D_x[\xi], && (13) \\ \eta^{(2)} &\equiv D_x[\eta^{(1)}] - y'\, D_x[\xi] = D_x\left[D_x[\eta] - y'\, D_x[\xi]\right] - y'\, D_x[\xi]. && (14) \end{aligned}$$

Since X is a symmetry generator for the 2ODE $y'' = \phi(x, y, y')$ then the vector field $\hat{X}^{(1)}$ defined by

$$
\begin{aligned}
\hat{X}^{(1)} &\equiv X^{(1)} - \rho\, D_x = \\
&= \xi\, \partial_x + \eta\, \partial_y + \eta^{(1)}\, \partial_{y'} - \rho\, \partial_x - \rho\, y'\, \partial_y - \rho\, \phi\, \partial_{y'} = \\
&= (\xi - \rho)\, \partial_x + (\eta - \rho\, y')\, \partial_y + (\eta^{(1)} - \rho\, \phi)\, \partial_{y'} = \\
&= (\xi - \rho)\, \partial_x + (\eta - y'\rho)\, \partial_y + (D_x[\eta - y'\rho] - y'\, D_x[\xi - \rho])\, \partial_{y'}, \qquad (15)
\end{aligned}
$$

is also a symmetry generator. Choosing $\rho = \xi$ we obtain a symmetry generator $\overline{X}^{(1)}$ in the evolutionary form:

$$\overline{X}^{(1)} = \overline{\eta}\, \partial_y + D_x[\overline{\eta}]\, \partial_{y'}, \tag{16}$$

where $\overline{\eta} \equiv \eta - y'\xi$. So, we have that $\overline{X}^{(2)} = \overline{\eta}\, \partial_y + D_x[\overline{\eta}]\, \partial_{y'} + D_x^2[\overline{\eta}]\, \partial_{y''}$ and we can write (12) as

$$D_x^2[\overline{\eta}] - D_x[\overline{\eta}]\, \phi_{y'} - \overline{\eta}\, \phi_y = 0, \tag{17}$$

as we want to demonstrate.□

Now we can prove theorem 1:

Proof of Theorem 1: If the hypothesis of the theorem is fulfilled, then we have that (see section 2.1, eq.(8))

$$dI = R\,((\phi + S\, y')dx - S\, dy - dy')\,.$$

So, we can write $I_x = R\,(\phi + S\, y')$, $I_y = -S\, R$, $I_{y'} = -R$. Using the compatibility conditions ($I_{xy} - I_{yx} = 0, I_{xy'} - I_{y'x} = 0$ and $I_{yy'} - I_{y'y} = 0$), we get:

$$R_y\,(\phi + S\, y') + R\,(\phi_y + S_y\, y') + (S_x\, R + S\, R_x) = 0\,, \tag{18}$$

$$R_{y'}\,(\phi + S\, y') + R\,(\phi_{y'} + S_{y'}\, y' + S) + R_x = 0\,, \tag{19}$$

$$-(R_{y'}\, S + R\, S_{y'}) + R_y = 0\,. \tag{20}$$

Eq.(19) plus eq.(20) times y' results

$$R_x + y'\, R_y + \phi\, R_{y'} + R\,(\phi_{y'} + S) = 0\,, \tag{21}$$

and eq.(18) minus eq.(20) times ϕ results

$$S\,(R_x + y'\, R_y + \phi\, R_{y'}) + R\,(S_x + y'\, S_y + \phi\, S_{y'}) + R\,\phi_y = 0\,. \tag{22}$$

These equations can be written, respectively, as

$$D_x[R] + R\,(\phi_{y'} + S) = 0\,, \tag{23}$$

$$S\, D_x[R] + R\, D_x[S] + R\,\phi_y = 0\,. \tag{24}$$

where D_x is the operator defined in (10). Isolating $D_x[R]$ in eq.(23) and substituting in eq.(24) we have that the S-function must obey the following equation:

$$D_x[S] = S^2 + \phi_{y'}\, S - \phi_y\,. \tag{25}$$

Now, let $\overline{\eta}$ be an infinitesimal defining a Lie symmetry in the evolutionary form. Substituting $S = -\frac{D_x[\overline{\eta}]}{\overline{\eta}}$ in (25) we have

$$D_x\left[-\frac{D_x[\overline{\eta}]}{\overline{\eta}}\right] = \left(-\frac{D_x[\overline{\eta}]}{\overline{\eta}}\right)^2 + \phi_{y'}\left(-\frac{D_x[\overline{\eta}]}{\overline{\eta}}\right) - \phi_y \;\;\Rightarrow\;\; -\frac{D_x^2[\overline{\eta}]}{\overline{\eta}} = -\,\phi_{y'}\frac{D_x[\overline{\eta}]}{\overline{\eta}} - \phi_y\,. \tag{26}$$

By using lemma 1 we can verify that eq.(26) is an identity and the theorem is demonstrated. □

If the reader is familiar with the classical theory of ODEs he/she can note two interesting facts:

- Since the operator D_x represents (over the solutions of the 2ODE) the total derivative with respect to x (i.e., $\frac{d}{dx}$), the PDE for the S-function is formally a Riccati ODE[6].
- If we apply the transformation $u = -v'/v$ into the Riccati ODE $u' = u^2 + g(x)\,u - h(x)$ we get the linear 2ODE given by $v'' = g(x)\,v' + h(x)\,v$. This 2ODE, is formally analogous to the linear second order PDE (linear 2PDE) for $\overline{\eta}$.

In the next section we will use the relation (9) to construct a semi-algorithmic method to calculate the Lie symmetries of a rational 2ODE. This method can be applied even to the case where there are only dynamical symmetries.

3 A method to calculate the Lie symmetries

The procedure can be divided in two main parts: first, we use the method developed in [27] to calculate the S-functions associated with the 2ODE; then we use the relation (9) to calculate the symmetries.

3.1 Calculating the S-functions

Based on theorem 2 of [27] we have the following result[7]:

Theorem 2: *Let*

$$y'' = \phi(x, y, y') = \frac{M(x, y, y')}{N(x, y, y')}, \tag{27}$$

be a rational 2ODE (i.e., M and N are polynomial functions of (x, y, y')), and Let $I(x, y, y')$ be an elementary first integral of it. Then, there is a S-function (S) associated with the 2ODE (27) such that:

(i) S is an algebraic function of (x, y, y').

[6] A Riccati ODE can be generally written as $u' = f(x)\,u^2 + g(x)\,u + h(x)$.

[7] For a proof, please see [27]

(*ii*) *The polynomial that defines the algebraic function* S *is an eigenpolynomial of the operator*
$\mathcal{D} \equiv (N)\, D + \left(N^2\, S^2 + (N\, \partial_{y'} M - M\, \partial_{y'} N)\, S + N\, \partial_y M - M\, \partial_y N\right) \partial_S,$
where $D \equiv N\, D_x$ *(*D_x *is the operator defined by (10)).*

The claiming (*ii*) is simply a more formal way of saying that the S-function must obey (25). Besides, it turns clearer what to do to obtain the S-function: we 'only' have to calculate the eigenpolynomials (the Darboux polynomials) of the operator $\mathcal{D}$. Each eigenpolynomial defines an algebraic function of (x, y, y') which is a S-function associated with the 2ODE (27). The emphasis on the word *only* is just to remember that calculating the eigenpolynomials of the operator $\mathcal{D}$ may not be an easy task. But, although the procedure may be hard to apply, it is of an algorithmic nature.

In this paper we want to deal with 2ODEs integrable by quadratures (i.e., presenting Liouvillian[8] first integrals). Since we do not know the general form of the S-funtions for this case, we will restrict ourselves to the case where the 2ODE presents rational S-funtions. We will show that even with this restriction, we can find symmetries in a lot of interesting cases.

3.2 Finding the symmetries

Once we have found the rational S-functions associated with the 2ODE (27), we will use the relation $D_x[\overline{\eta}]/\overline{\eta} = -S$ to obtain a Lie symmetry. As we have mentioned in the last section, we will restrict ourselves to the case where the 2ODE (27) presents rational S-funtions. For this case we were able to prove the following result (this paper's second goal):

Theorem 3: *Let* $y'' = \phi(x, y, y') = M/N$ *(*M *and* N *polynomials of* (x, y, y') *) be a rational 2ODE presenting two independent Liouvillian first integrals* $I_1(x, y, y')$ *and* $I_2(x, y, y')$*. If this 2ODE has two rational* S*-functions* S_1 *and* S_2 *associated with it such that* $dI_1 = R_1((\phi + S_1\, y')dx - S_1\, dy - dy')$ *and* $dI_2 = R_2((\phi + S_2\, y')dx - S_2\, dy - dy')$*, then the following statements hold:*

(*i*) *There exists two independent Lie symmetries in the evolutionary form (16) such that the infinitesimals* $\overline{\eta}_1$ *and* $\overline{\eta}_2$ *are Darboux functions of* (x, y, y')*, i.e., the infinitesimals* $\overline{\eta}_1$ *and* $\overline{\eta}_2$ *have the form*

$$\overline{\eta}_1 \;=\; e^{\frac{A_1}{B_1}} \prod_i {p_1}_i^{\,{c_1}_i}, \tag{28}$$

$$\overline{\eta}_2 \;=\; e^{\frac{A_2}{B_2}} \prod_j {p_2}_j^{\,{c_2}_j}. \tag{29}$$

where A_1, B_1, A_2, B_2, *the* ${p_1}_i$ *and the* ${p_2}_j$ *are polynomial functions of* (x, y, y') *and the* ${c_1}_i$ *and* ${c_2}_j$ *are constants.*

(*ii*) *The* ${p_1}_i$ *and the* ${p_2}_j$ *are irreductible eigen-polynomials (Darboux polynomials) of the operator* D *(defined by* $N\, D_x$ *) or are irreductible factors of the denominators of* S_1, S_2 *, respectively.*

[8]For a formal definition of Liouvillian function, please see [28].

(iii) The irreductible factors of the polynomials B_1 and B_2 are irreductible eigen-polynomials (Darboux polynomials) of the operator D or are irreductible factors of the denominators of S_1, S_2, respectively.

In order to prove this theorem we will need some results:

Proposition 1: *Let $y'' = \phi(x, y, y')$ be a 2ODE presenting a first integral $I(x, y, y')$. If $S(x, y, y')$ is a S-function associated with it, then the 1ODE defined by*

$$\frac{dv}{du} = -S(a_1, u, v) \tag{30}$$

has $I(a_1, u, v) = C$ (where C is a constant) as its general solution.

Proof of Proposition 1: $I(a_1, u, v) = C$ is a general solution of the 1ODE $dv/du = -S(a_1, u, v)$ if and only if

$$D_u[I] = 0, \tag{31}$$

where $D_u \equiv \partial_u - S\,\partial_v$. The condition (31) means that $\partial_u I - S\,\partial_v I = 0$, i.e., $S = I_u/I_v$. Consider that the hypothesis of proposition 1 are satisfied. To prove proposition 1 we only have to prove that $S = I_y/I_{y'}$. According to (8) we can write $dI = R((\phi + S\,y')dx - S\,dy - dy')$, leading to $I_y = -R\,S$ and $I_{y'} = -R$. So, $S = I_y/I_{y'}$. $\square$

Using similar reasoning we can enunciate the following[9]:

Proposition 2: *Let $y'' = \phi(x, y, y')$ be a 2ODE presenting a first integral $I(x, y, y')$. If $S(x, y, y')$ is a S-function associated with it, then the 1ODE defined by*

$$\frac{dv}{dt} = \phi(t, a_2, v) + v\,S(t, a_2, v) \tag{32}$$

has $I(t, a_2, v) = C$ (where C is a constant) as its general solution.

Proposition 3: *Let $y'' = \phi(x, y, y')$ be a 2ODE presenting a first integral $I(x, y, y')$. If $S(x, y, y')$ is a S-function associated with it, then the 1ODE defined by*

$$\frac{du}{dt} = \frac{\phi(t, u, a_3) + a_3\,S(t, u, a_3)}{S(t, u, a_3)} \tag{33}$$

has $I(t, u, a_3) = C$ (where C is a constant) as its general solution.

Definition 3: *The three 1ODEs (30), (32) and (33) will be called* auxiliary 1ODEs *of the 2ODE (27) associated with the S-function $S(x, y, y')$.*

In what follows, we will refer to (30) as 1ODE A1, to (32) as 1ODE A2 and to (33) as 1ODE A3. This concept (of auxiliary 1ODE) will be useful to enunciate the following result. Note that the general solution of the auxiliary 1ODEs (A1, A2 and A3) are defined by the function I that is a first integral of the 2ODE. The following result establishes another important link between the 2ODE and the auxiliary 1ODEs.

[9]The proofs of propositions 2 and 3 are analogous to that of proposition 1.

Proposition 4: *Let $y'' = \phi(x, y, y')$ be a 2ODE presenting a Lie symmetry that can be written (in the evolutionary form) as*

$$\overline{X}^{(1)} = \overline{\eta}(x, y, y')\,\partial_y + D_x[\overline{\eta}(x, y, y')]\,\partial_{y'}. \tag{34}$$

Let $\overline{\zeta} \equiv D_x[\overline{\eta}]$, then the operator $Y \equiv \overline{\eta}(a_1, u, v)\,\partial_u + \overline{\zeta}(a_1, u, v)\,\partial_v$ is a generator for a Lie point symmetry of an auxiliary 1ODE A1 (30).

Proof of Proposition 4: The condition for a vector field $Y \equiv \overline{\eta}(a_1, u, v)\,\partial_u + \overline{\zeta}(a_1, u, v)\,\partial_v$ to be a symmetry vector of the 1ODE (30) is that its commutator with the operator $D_u \equiv \partial_u - S\,\partial_v$ is proportional to D_u. So we have to prove that $[Y, D_u]$ is proportional to D_u. We have that:

$$\begin{aligned} [Y, D_u] &= -(\overline{\eta}\,S_u + \overline{\zeta}\,S_v)\,\partial_v - (\overline{\eta}_u - S\,\overline{\eta}_v)\,\partial_u - (\overline{\zeta}_u - S\,\overline{\zeta}_v)\,\partial_v = \\ &= -D_u[\overline{\eta}]\,\partial_u - (\overline{\eta}\,S_u + \overline{\zeta}\,S_v + \overline{\zeta}_u - S\,\overline{\zeta}_v)\,\partial_v. \end{aligned} \tag{35}$$

Since $S = -D_x[\overline{\eta}]/\overline{\eta} = -\overline{\zeta}/\overline{\eta}$ we can write

$$\begin{aligned} [Y, D_u] &= -D_u[\overline{\eta}]\,\partial_u - \left(\overline{\eta}\,\frac{\overline{\eta}\,\overline{\zeta}_u + \overline{\zeta}\,\overline{\eta}_u}{\overline{\eta}^2} + \overline{\zeta}\,\frac{\overline{\eta}\,\overline{\zeta}_v + \overline{\zeta}\,\overline{\eta}_v}{\overline{\eta}^2} + \overline{\zeta}_u - S\,\overline{\zeta}_v\right)\partial_v \\ &= -D_u[\overline{\eta}]\,\partial_u - (-S\,D_u[\overline{\eta}])\,\partial_v, \end{aligned} \tag{36}$$

implying that $[Y, D_u] = -D_u[\overline{\eta}]\,D_u$. $\square$

Proposition 5: Let $y'' = \phi(x, y, y')$ be a 2ODE presenting two independent first integrals $I_1(x, y, y')$ and $I_2(x, y, y')$. If $S_1 = I_{1y}/I_{1z}$ and $S_2 = I_{2y}/I_{2z}$ are two S-functions associated with it such that $dI_1 = R_1((\phi + S_1\,y')dx - S_1\,dy - dy')$ and $dI_2 = R_2((\phi + S_2\,y')dx - S_2\,dy - dy')$, then there are infinitesimals for Lie symmetries in the evolutionary form, $\overline{\eta}_1$ and $\overline{\eta}_2$, (such that $-D_x[\overline{\eta}_1]/\overline{\eta}_1 = S_1$ and $-D_x[\overline{\eta}_2]/\overline{\eta}_2 = S_2$) given by

$$\overline{\eta}_1 = \frac{1}{(S_1 - S_2)\,R_2}, \tag{37}$$

$$\overline{\eta}_2 = \frac{1}{(S_2 - S_1)\,R_1}. \tag{38}$$

Proof of Proposition 5: Consider that the hypothesis of the proposition are fulfilled. Let the operators $X_1 \equiv \overline{\eta}_1\,\partial_y + D_x[\overline{\eta}_1]\,\partial_{y'}$ and $X_2 \equiv \overline{\eta}_2\,\partial_y + D_x[\overline{\eta}_2]\,\partial_{y'}$ be Lie symmetries of the 2ODE $y'' = \phi(x, y, y')$ in the evolutionary form. Without loss of generality we can choose $\overline{\eta}_1$ and $\overline{\eta}_2$ such that $X_1[I_1] = X_2[I_2] = 0$ and $X_1[I_2] = X_2[I_1] = 1$. Using $X_1[I_1] = 0$ and $X_1[I_2] = 1$ we have

$$\begin{aligned} \overline{\eta}_1\,I_{1y} + D_x[\overline{\eta}_1]\,I_{1y'} = 0, \\ \overline{\eta}_1\,I_{1y} + D_x[\overline{\eta}_1]\,I_{2y'} = 1, \end{aligned}$$

implying that

$$\overline{\eta}_1 = \frac{-I_{1y'}}{I_{1y}\,I_{2y'} - I_{1y'}\,I_{2y}} = \frac{-1}{\frac{I_{1y}}{I_{1y'}}\,I_{2y'} - I_{2y}} = \frac{-1}{I_{2y'}\left(\frac{I_{1y}}{I_{1y'}} - \frac{I_{2y}}{I_{2y'}}\right)}. \tag{39}$$

Since $I_{2y'} = -R_2$, $I_{1y}/I_{1y'} = S_1$ and $I_{2y}/I_{2y'} = S_2$ we can write $\overline{\eta}_1 = 1/((S_1 - S_2)\,R_2)$ and we can deduce the format (38) for $\overline{\eta}_2$ in an analogous way. $\square$

Theorem 4: *If a rational 1ODE written in the form $y' = M(x,y)/N(x,y)$ (M and N are polynomials with no common factor) has a general solution of the form $I(x,y) = C$ where I is a Liouvillian function of its arguments and C is an arbitrary constant, then the 1-form $M\,dx - N\,dy$ presents an integrating factor R of the form*

$$R = e^{r_0} \prod_i {p_i}^{c_i}, \tag{40}$$

where r_0 is a rational function of (x,y), the p_i and the factors of the denominator of r_0 are irreductible Darboux polynomials of the operator $N\,\partial_x + M\,\partial_y$ and the c_i are constants.

For a proof of theorem 4 see [31, 32, 33, 34] or [35, 36].

Now we can prove theorem 3. The key point in the following demonstration is the fact that the general solutions of the auxiliary 1ODEs associated with S are defined by the same function I that is a first integral of the rational 2ODE $y'' = \phi(x,y,y') = M/N$.

Proof of Theorem 3: First of all let's stablish some notation: we will write $S_1 = P_1/Q_1$ and $S_2 = P_2/Q_2$, where P_1, Q_1, P_2 and Q_2 are polynomials of (x,y,y') and the pairs P_1, Q_1 and P_2, Q_2 do not have any common factors.

Consider now that the hypothesis of the theorem are fulfilled. From proposition 1, we have that $I_1(a_1,u,v) = C$ is a general Liouvillian solution of the rational auxiliary 1ODE A1 associated with S_1

$$\frac{dv}{du} = -S_1(a_1,u,v) = -\frac{P_1(a_1,u,v)}{Q_1(a_1,u,v)}. \tag{41}$$

Then (from theorem 4) the 1-form $P_1(a_1,u,v)\,du + Q_1(a_1,u,v)\,dv$ presents an integrating factor of the form (40). Let $R_{[A1,S1]}(a_1,u,v)$ be such an integrating factor. Since

$$\begin{aligned} dI_1(x,y,y') &= R_1\left((\phi + S_1\,y')dx - S_1\,dy - dy'\right) \\ &= \frac{R_1}{Q_1\,N}\left((M\,Q_1 + P_1\,N\,y')dx - P_1\,N\,dy - Q_1\,N\,dy'\right), \end{aligned} \tag{42}$$

we can see that

$$\frac{\partial I_1}{\partial y}(x,y,y') = -\frac{R_1(x,y,y')\,P_1(x,y,y')}{Q_1(x,y,y')} \quad \text{and} \quad \frac{\partial I_1}{\partial y'}(x,y,y') = -R_1(x,y,y')$$

implying that

$$\frac{\partial I_1}{\partial u}(a_1,u,v) = -\frac{R_1(a_1,u,v)\,P_1(a_1,u,v)}{Q_1(a_1,u,v)} \quad \text{and} \quad \frac{\partial I_1}{\partial v}(a_1,u,v) = -R_1(a_1,u,v).$$

Therefore, $-R_1(a_1,u,v)/Q_1(a_1,u,v)$ is also an integrating factor for the 1-form $P_1(a_1,u,v)\,du + Q_1(a_1,u,v)\,dv$. Then, we can write $\frac{R_1}{Q_1} = \mathcal{F}_1(I_1)\,R_{[A1,S1]}$ (where

$\mathcal{F}_1$ is a function of I_1) or $\frac{R_1}{Q_1} = k_1\, R_{[A1,S1]}$ (where k_1 is a constant) leading to $R_1 = \mathcal{F}_1(I_1)\, R_{[A1,S1]}\, Q_1$ (or $R_1 = k_1\, R_{[A1,S1]}\, Q_1$). So, $R_{[A1,S1]}(x,y,y')\, Q_1(x,y,y')$ is an integrating factor for the 1-form $(\phi +\, S_1\, y')dx - S_1\, dy - dy'$.

Besides, from proposition 2 we have that $I_1(t, a_2, v) = C$ is a general Liouvillian solution of the rational auxiliary 1ODE A2 associated with S_1

$$\frac{dv}{dt} = \phi(t,a_2,v) + v\, S_1(t,a_2,v) = \frac{Q_1(t,a_2,v)\, M(t,a_2,v) + v\, P_1(t,a_2,v)\, N(t,a_2,v)}{Q_1(t,a_2,v)\, N(t,a_2,v)}. \tag{43}$$

Then (from theorem 4) the 1-form $-(Q_1\, M + v\, P_1\, N)\, dt + Q_1\, N\, dv$ presents an integrating factor of the form (40). Let $R_{[A2,S1]}(t,a_2,v)$ be such an integrating factor. From (42) and using a completely analogous reasoning of what we have shown above, we can conclude that $R_1(t,a_2,v)/\,(Q_1(t,a_2,v)\, N(t,a_2,v))$ is also an integrating factor for the 1-form $-(Q_1\, M + v\, P_1\, N)\, dt + Q_1\, N\, dv$. Then, we can write $\frac{R_1}{Q_1\, N} = \mathcal{F}_2(I_1)\, R_{[A2,S1]}$ (where $\mathcal{F}_2$ is a function of I_1) or $\frac{R_1}{Q_1\, N} = k_2\, R_{[A2,S1]}$ (where k_2 is a constant) leading to $R_1 = \mathcal{F}_2(I_1)\, R_{[A2,S1]}\, Q_1\, N$ (or $R_1 = k_2\, R_{[A2,S1]}\, Q_1\, N$). So, $R_{[A2,S1]}(x,y,y')\, Q_1(x,y,y')\, N(x,y,y')$ is an integrating factor for the 1-form $(\phi +\, S_1\, y')dx - S_1\, dy - dy'$.

Now let's see: since $R_{[A1,S1]}(x,y,y')\, Q_1(x,y,y')$ and $R_{[A2,S1]}(x,y,y')\, Q_1(x,y,y')$ $N(x,y,y')$ are integrating factors for the 1-form $(\phi +\, S_1\, y')dx - S_1\, dy - dy'$, then we can write $R_{[A1,S1]}(x,y,y')Q_1(x,y,y') = \mathcal{F}(I_1)\, R_{[A2,S1]}(x,y,y')\, Q_1(x,y,y')N(x,y,y')$ ($\mathcal{F}$ is a function of I_1) or $R_{[A1,S1]}(x,y,y')\, Q_1(x,y,y') = k_1\, R_{[A2,S1]}(x,y,y')\, Q_1(x,y,$ $y')\, N(x,y,y')$ (k is a constant).

- First possibility: $R_{[A1,S1]}(x,y,y') = k\, R_{[A2,S1]}(x,y,y')\, N(x,y,y')$. Let's remember that the polynomials forming $R_{[A1,S1]}(x,y,y')$ are polynomials of (y,y') and the polynomials forming $R_{[A2,S1]}(x,y,y')$ are polynomials of (x,y') . Therefore, we can conclude that $R_{[A1,S1]}(x,y,y')$ and $R_{[A2,S1]}(x,y,y')$ are of the form $\exp[r_0]\, \prod_i\, {p_i}^{c_i}$, where the p_i are polynomials of (x,y,y') and r_0 is a rational function of (x,y,y') .
- Second possibility: $R_{[A1,S1]}(x,y,y') = \mathcal{F}(I_1)\, R_{[A2,S1]}(x,y,y')\, N(x,y,y')$. For the case where I_1 is an elementary function of (x,y,y') , from the results presented in [29] we can conclude that $R_{[A1,S1]}(x,y,y')$ and $R_{[A2,S1]}(x,y,y')$ are of the form $\prod_i\, {p_i}^{c_i}$ and use the same reasoning above to show that the p_i are polynomials of (x,y,y') .

  Now, let's suppose that I_1 is a non elementary Liouvillian function of (x,y,y') . Since $R_{[A1,S1]}(x,y,y')$ and $R_{[A2,S1]}(x,y,y')$ are elementary functions, the only possibility for the relation $R_{[A1,S1]}(x,y,y') = \mathcal{F}(I_1)\, R_{[A2,S1]}(x,y,y')\, N(x,y,y')$ to be true is that $\mathcal{F}(I_1)$ is the constant function, i.e., $\mathcal{F}(I_1) = k$. So, we are again in the first possibility and we have that $R_{[A1,S1]}(x,y,y')$ and $R_{[A2,S1]}(x,y,y')$ are of the form $\exp[r_0]\, \prod_i\, {p_i}^{c_i}$, where the p_i are polynomials of (x,y,y') and r_0 is a rational function of (x,y,y') .

Now, consider the auxiliary rational 1ODEs A1 and A2 associated with S_2 :

$$\frac{dv}{du} = -S_2(a_1,u,v) = -\frac{P_2(a_1,u,v)}{Q_2(a_1,u,v)}, \tag{44}$$

$$\frac{dv}{dt} = \phi(t,a_2,v) + v\,S_2(t,a_2,v) = \frac{Q_2(t,a_2,v)\,M(t,a_2,v) + v\,P_2(t,a_2,v)\,N(t,a_2,v)}{Q_2(t,a_2,v)\,N(t,a_2,v)}. \tag{45}$$

From theorem 4 (and using propositions 2 and 3 again), we can infer that the 1-forms $P_2(a_1,u,v)\,du + Q_2(a_1,u,v)\,dv$ and $-(Q_2(t,a_2,v)\,M(t,a_2,v) + v\,P_2(t,a_2,v)\,N(t,a_2,v))\,dt + Q_2(t,a_2,v)\,N(t,a_2,v)\,dv$ present integrating factors of the form (40). Let $R_{[A1,S2]}(a_1,u,v)$ and $R_{[A2,S2]}(t,a_2,v)$ be, respectively, these integrating factors. Using the same reasoning above we can conclude that (analogously as above):

- $-R_{[A1,S2]}(x,y,y')\,Q_2(x,y,y')$ is an integrating factor for the 1-form $(\phi + S_2\,y')dx - S_2\,dy - dy'$.
- $R_{[A2,S2]}(x,y,y')\,Q_2(x,y,y')\,N(x,y,y')$ is an integrating factor for the 1-form $(\phi + S_2\,y')dx - S_2\,dy - dy'$.
- $R_{[A1,S2]}(x,y,y')$ and $R_{[A2,S2]}(x,y,y')$ are of the form $\exp[r_0]\prod_i {p_i}^{c_i}$, where the p_i are polynomials of (x,y,y') and r_0 is a rational function of (x,y,y').

Therefore, from these results and the analogous above, we can afirm that there exists two independent Liouvillian first integrals $\overline{I}_1$ and $\overline{I}_2$ such that

$$d\overline{I}_1(x,y,y') = \overline{R}_1\left((\phi + S_1\,y')dx - S_1\,dy - dy'\right), \tag{46}$$

$$d\overline{I}_2(x,y,y') = \overline{R}_2\left((\phi + S_2\,y')dx - S_2\,dy - dy'\right), \tag{47}$$

where $\overline{R}_1 \equiv -R_{[A1,S1]}\,Q_1$ and $\overline{R}_2 \equiv -R_{[A1,S2]}\,Q_2$ are of the form $e^{A/B}\prod_i {p_i}^{c_i}$ (A, B, p_i polynomials in (x,y,y') and c_i constants).

We can now prove (i). From (46), (47) and proposition 5 we can conclude (since S_1 and S_2 are rational functions of (x,y,y')) that there are infinitesimals $\overline{\eta}_1$ and $\overline{\eta}_2$ of the form (28) and (29).

In order to prove (ii) and (iii) let's first note that

$$-\frac{D_x[\overline{\eta}_1]}{\overline{\eta}_1} = \frac{P_1}{Q_1} \;\Rightarrow\; Q_1\frac{D[\overline{\eta}_1]}{\overline{\eta}_1} = -N\,P_1\,, \tag{48}$$

$$-\frac{D_x[\overline{\eta}_2]}{\overline{\eta}_2} = \frac{P_2}{Q_2} \;\Rightarrow\; Q_2\frac{D[\overline{\eta}_2]}{\overline{\eta}_2} = -N\,P_2\,, \tag{49}$$

i.e., $Q_1\frac{D[\overline{\eta}_1]}{\overline{\eta}_1}$ and $Q_2\frac{D[\overline{\eta}_2]}{\overline{\eta}_2}$ are polynomials in (x,y,y'). So, we have the situation: the infinitesimals $\overline{\eta}_1$ and $\overline{\eta}_2$ are Darboux functions such that $\frac{\mathcal{D}_1[\overline{\eta}_1]}{\overline{\eta}_1} = pol_1$ and $\frac{\mathcal{D}_2[\overline{\eta}_2]}{\overline{\eta}_2} = pol_2$, where $\mathcal{D}_1 \equiv Q_1\,D$ and $\mathcal{D}_2 \equiv Q_2\,D$. Finally, from the main result presented in [33], we can directly conclude (ii) and (iii). $\square$

The procedure to calculate the symmetries in this case is completely analogous to that for calculate the integrating factors of rational 1ODEs with Liouvillian solution (see [34]): briefly, we calculate the Darboux polynomials (up to a certain degree) of the operator D. Then we substitute the candidates for $\overline{\eta}$ in the equation

$$Q\frac{D[\overline{\eta}]}{\overline{\eta}} = -N\,P\,, \tag{50}$$

and we solve the resulting linear equations for the constants c_i.

4 Examples

In this section we will present two examples where we can use the semi-algorithm presented above to calculate the Lie symmetries. In the first example we will show a 2ODE with one point symmetry and one Darboux dynamical symmetry to ilustrate theorem 3. This symmetry can be found by our method – a process devoided of any guess. We will use this example to show our procedure in action, i.e., we will provide comments about the calculations to clearer the steps of the method. In the second example we use a 2ODE not presenting Lie point symmetries but with two rational dynamical symmetries. The good point to be noted is that, for our kind of process, the existence or not of point symmetries makes no difference.

4.1 First example

Consider the 2ODE given by

$$y'' = \frac{2\,y - 3\,zy + z^2y - zx + z^2x}{y(y-x)}. \tag{51}$$

To use the procedure explained in the section 3 we have, first, to obtain the operators D and $\mathcal{D}$. They are given by

$$D = (x\,y - y^2)\frac{\partial}{\partial x} + (x\,y\,z - y^2z)\frac{\partial}{\partial y} + (3\,y\,z - yz^2 - xz^2 - 2\,y + x\,z)\frac{\partial}{\partial z}, \tag{52}$$

$$\begin{aligned}\mathcal{D} = {} & (y^4 + y^2x^2 - 2\,y^3x)\frac{\partial}{\partial x} + (-2\,zy^3x + zy^4 + zy^2x^2)\frac{\partial}{\partial y} + \\ & (yzx^2 - yx^2z^2 + 2\,y^3 + 2\,y^2zx + y^3z^2 - 3\,zy^3 - 2\,xy^2)\frac{\partial}{\partial z} + \\ & \Big((y^2x^2 + y^4 - 2\,y^3x)\,s^2 + (-2\,zyx^2 + 2\,zy^3 + 2\,xy^2 + yx^2 - 3\,y^3)\,s \\ & -z^2x^2 + zx^2 + z^2y^2 - 2\,zyx + 2\,y^2 - 3\,zy^2 + 2\,z^2yx\Big)\frac{\partial}{\partial s}.\end{aligned} \tag{53}$$

The second step is to calculate the Darboux polynomials of the operator $\mathcal{D}$. We can find two Darboux polynomials:

$$p_{s1} = (z-1)(y+x) + s\,y\,(x-y), \tag{54}$$

$$p_{s2} = x\,(-2\,y + zy + zx)(z-1) + s\,y\,(x-y)(zx-y). \tag{55}$$

The S-functions are the solutions of *Darboux polynomial of* $\mathcal{D} = 0$. From (54) and (55) we can obtain two rational S-functions given by

$$S1 = \frac{(z-1)\,(y+x)}{y\,(x-y)}, \tag{56}$$

$$S2 = \frac{x\,(-2\,y + zy + zx)\,(z-1)}{(x-y)\,(zx-y)\,y}. \tag{57}$$

The next step is to calculate the Darboux polynomials of the D operator and the corresponding cofactors to build the infinitesimals. They are

$$p_1 = y \quad \Rightarrow \quad q_1 = (x-y)\,z, \tag{58}$$

$$p_2 = x - y \quad \Rightarrow \quad q_2 = -\left(z - 1\right)\, y, \tag{59}$$

$$p_3 = z - 1 \quad \Rightarrow \quad q_3 = 2\, y - zy - zx. \tag{60}$$

Using these Darboux polynomials and looking at (50), we can use the semi-algorithm described in [34, 37] to calculate the infinitesimals:

$$\overline{\eta}_1 = -\frac{\left(x - y\right)^2 e^{\frac{x-y}{(z-1)y}}}{y}, \tag{61}$$

$$\overline{\eta}_2 = \frac{\left(zx - y\right)\left(x - y\right)^2}{\left(z - 1\right) y}. \tag{62}$$

We can use the relations $X_1[I_1] = 0\,,\; X_2[I_1] = 1\,,\; D[I_1] = 0$ and solve them to $I_{1x}\,,\; I_{1y}$ and $I_{1z}\,.$ We can also use the relations $X_1[I_2] = 1\,,\; X_2[I_2] = 0\,,\; D[I_2] = 0$ and solve them to $I_{2x}\,,\; I_{2y}$ and $I_{2z}\,.$ From the derivatives we can integrate and obtain I_1 and I_2:

$$I_1 = \frac{y\left(z - 1\right)}{\left(y - x\right)^2}, \tag{63}$$

$$I_2 = \int \frac{e^{\frac{y-x}{y(z-1)}}\left(-2\, y + zy + zx\right) y}{\left(y - x\right)^3}\, dx. \tag{64}$$

4.2 second example

It can be proved (see [19]) that the 2ODE

$$y'' = -\frac{x^2 + 4\, y^4 + 2\, y^2}{4\, y^3} \tag{65}$$

has no Lie point symmetries. By doing the procedure explained in the section 3 we have:
The operator D and $\mathcal{D}$ are given by

$$D = 4\, y^3 \frac{\partial}{\partial x} + 4\, y^3 z \frac{\partial}{\partial y} + \left(-x^2 - 4\, y^4 - 2\, y^2\right) \frac{\partial}{\partial z}, \tag{66}$$

$$\mathcal{D} = 16\, y^6 \frac{\partial}{\partial x} + 16\, y^6 z \frac{\partial}{\partial y} + 4\, y^2 \left(-4\, y^5 - 2\, y^3 - yx^2\right) \frac{\partial}{\partial z} +$$
$$+4\, y^2 \left(-3\, x^2 + 4\, y^4 s^2 - 2\, y^2 + 4\, y^4\right) \frac{\partial}{\partial s}. \tag{67}$$

We can find two Darboux polynomials of the operator $\mathcal{D}$:

$$p_{s1} = x + yz + S\, y^2, \tag{68}$$

$$p_{s2} = 4\, y^4 x + 8\, y^5 z + 4\, y^2 x + 4\, y^3 z + x^3 + 2\, x^2 yz + 2\, i\, (x^2 y^2 + 4\, y^6) +$$
$$+4\, S\, y^3 \left(-y + zx + 2\, yz^2 + 2\, i\, y^2 z\right). \tag{69}$$

From (68) and (69) we can obtain two rational S-functions given by

$$S1 = -\frac{x+yz}{y^2}, \tag{70}$$

$$S2 = \frac{4\,y^4x + 8\,y^5z + 4\,y^2x + 4\,y^3z + x^3 + 2\,x^2yz + 2\,i\,(x^2y^2 + 4\,y^6)}{4\,(-y + zx + 2\,yz^2 + 2\,i\,y^2z)\,y^3}. \tag{71}$$

The Darboux polynomials of the D operator and the corresponding cofactors are

$$p_1 = y \quad \Rightarrow \quad q_1 = 4\,y^2z, \tag{72}$$

$$p_2 = x + 2\,yz + 2\,i\,y^2 \quad \Rightarrow \quad q_2 = -2\,y(x - 2\,yz - 2\,i\,y^2), \tag{73}$$

$$p_3 = x + 2\,yz - 2\,i\,y^2 \quad \Rightarrow \quad q_3 = -2\,y(x - 2\,yz + 2\,i\,y^2). \tag{74}$$

Using these Darboux polynomials, we can use (50) to calculate the infinitesimals

$$\overline{\eta}_1 = \frac{4\,y^3}{(x + 2\,yz - 2\,i\,y^2)\,(x + 2\,yz + 2\,i\,y^2)}, \tag{75}$$

$$\overline{\eta}_2 = -\frac{-y + zx + 2\,yz^2 + 2\,i\,y^2z}{4(x + 2\,yz + 2\,i\,y^2)}. \tag{76}$$

We can use the relations $X_1[I_1] = 0$, $X_2[I_1] = 1$, $D[I_1] = 0$, $X_1[I_2] = 1$, $X_2[I_2] = 0$, $D[I_2] = 0$ to obtain I_1 and I_2:

$$I_1 = 4\,x - 2\,i\,\ln\left(\frac{x + 2\,yz - 2\,i\,y^2}{x + 2\,yz + 2\,i\,y^2}\right), \tag{77}$$

$$I_2 = \frac{1}{2}\,y^2 + \frac{1}{2}\,i\,x + \ln(y) - \frac{1}{2}\,\ln(x + 2\,yz + 2\,i\,y^2) - \frac{1}{8}\,\frac{x^2}{y^2} + \frac{1}{2}\,z^2. \tag{78}$$

In [19] Muriel and Romero were capable of dealing with this 1ODE (65) by developing the concept of $C^\infty(M^{(1)})$-symmetries – vector fields (also called λ-symmetries) that are neither Lie symmetries nor Lie-Bäcklund symmetries. The determining equations for the components of these vector fields depend on an arbitrary function λ, which can be chosen in order to simplify the process of solution of the determining equations. For this 1ODE they obtained $\lambda = x/u^2$ and $v = x\frac{\partial}{\partial x}$ for the λ-symmetry.

In our process (shown above) the main cost (computational cost) is to calculate the Darboux polynomials of the operator $\mathcal{D}$ (followed by calculating the Darboux polynomials of the operator D). The good news are that the process is entirely computational, i.e., devoid of any guess.

5 Conclusion

Although the Lie symmetry method is the most powerful method for solving, reducing and studying dynamical systems since its appearance (end of XIXth century), there are (still today) a bunch of open questions. The main 'gap' is the absence of an algorithm to calculate the symmetry vector fields. The things go worse when

the dynamical system under study does not present Lie point symmetries (in which case we can not even count with a systematic method to deal with the problem).

In this paper we have proposed a semi-algorithm to find Lie symmetries for a rational 2ODE. We began by showing a deep connection between the S-functions (see section 2.1 and [1, 27]) of a 2ODE and its symmetries in the evolutionary form. Then, restricting ourselves to the case where the S-functions are rational and the first integrals are Liouvillian, we could develop a semi-algorithm to calculate the symmetries.

The great advantage of our method is that it converts the search for symmetries into (essentially) searching the Darboux polynomials of the polynomial differential linear operators $\mathcal{D}$ and D (in other words, into solving second degree algebraic equations) – a semi-algorithmic procedure. The disadvantage is that, as the degree of the Darboux polynomials grows, the computational coast increases 'a lot' and, sometimes, turns the entire process inviable.

Our method is not general since it is limited to the cases where the S-functions are rational (see section 3). However, from a practical point of view, this restriction does not seems to be a great one: all the non-linear rational 2ODEs (that we have analysed) with two Liouvillian first integrals, presented rational S-functions.

In regard to future work, there are some main directions (open questions):

- What is the general form of the S-functions for a 2ODE presenting two Liouvillian first integrals? (and only one Liouvillian first integral?)
- What about NODEs?
- What happens if we consider elementary functions present in the ODE?
- What about systems of ODEs?
- Etc...

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